# Lecture 05: Repeated Squaring

# Recall

- Let $(G, \circ)$ be a group with generator $g$
- We define $g^0 = e$, where $r \in G$ is the identity element of $G$
- We define $g^i = \overbrace{g \circ g \circ \cdots \circ g}^{i\text{-times}}$
- For example, the group $(\mathbb{Z}_7^*, \times)$ is generated by 3 but not 2

# Motivation of Efficient Algorithm to Compute Exponentiation

- Suppose $p$ is a prime number that is represented using 1000-bits
- Note that the number $p$ is in the range $[2^{999}, 2^{1000})$. We shall summarize this by stating that $p$ is roughly (in the order of) $2^{1000}$.
- Suppose we are interested to work on the field $(\mathbb{Z}_p^*, \times)$ with generator $g$
- Given input $i \in \{0, 1, \ldots, p-1\}$, we are interested in computing $g^i \in \mathbb{Z}_p^*$

# First Attempt

Exp ($i$):

1. prod $= e$
2. For index in the range $\{1, ..., i\}$:
    1. prod $=$ prod $\circ\, g$
3. Return prod

- Note that this algorithm runs the inner loop $i$ times. The number $i$ can take values $\{0, 1, \ldots, p-2\}$. For example, if $i \geqslant 2^{500}$ then the algorithm will run the inner loop more than the number of atoms in the universe. Effectively, the algorithm is useless
- The algorithm takes $O(i)$ run-time. The size of the input $i$ is $\log i$. So, this algorithm is an exponential time algorithm

# Second Attempt I

Exp ($i$):

1. If $i = 0$: Return $e$
2. If $i$ is even:
   1. $\alpha = \text{Exp}(i/2)$
   2. Return $\alpha \circ \alpha$
3. If $i$ is odd:
   1. $\alpha = \text{Exp}((i-1)/2)$
   2. Return $\alpha \circ \alpha \circ g$

- Note that the argument to Exp becomes smaller by one-bit in recursive call. So, the algorithm performs (at most) 1000 recursive call. This is an efficient algorithm because it runs in time $O(\log i)$

**A Few Optimizations.**

- Testing whether $i$ is even or not can be performed by computing $i\&1$ (here, & is the bit-wise and of the binary representation of $i$ and 1
- Computing $(i/2)$ when $i$ is even, or computing $(i-1)/2$ when $i$ is odd can be achieved by $i \gg 1$ (that is, right-shift the binary representation of $i$ by one position)

# Second Attempt III

The code shall look as follows

Exp ($i$):

1. If $i = 0$: Return $e$
2. $j \gg 1$
3. If $(i\&1) == 0$:
    1. $\alpha = \text{Exp}(j)$
    2. Return $\alpha \circ \alpha$
4. else:
    1. $\alpha = \text{Exp}(j)$
    2. Return $\alpha \circ \alpha \circ g$

# Second Attempt IV

1. The algorithm makes recursive calls. Can we further optimize and avoid recursive function calls? That is, can we unroll the recursion into a for loop?

# Final Attempt I

In the following code, we assume that we represent the prime $p$ using $t$-bits. For example, we were considering $t = 1000$ in the ongoing example.

We perform a preprocessing step to compute the following global variables.

**Global Preprocessing.**

1. For index in the set $\{0, 1, \ldots, t-1\}$:
    1. If index $== 0$: $\alpha_{\text{index}} = g$ and $c_{\text{index}} = 1$
    2. Else: $\alpha_{\text{index}} = \alpha_{\text{index}-1} \circ \alpha_{\text{index}-1}$ and $c_{\text{index}} = (c_{\text{index}-1} \ll 1)$

- Note that $\alpha_{\text{index}} = g^{2^{\text{index}}}$, for all index $\in \{0, 1, \ldots, t-1\}$
- Further, note that $c_{\text{index}} = 2^{\text{index}}$, for all index $\in \{0, 1, \ldots, t-1\}$

# Final Attempt II

We shall use the preprocessed data to compute the exponentiation

Exp ($i$):

1. prod $= e$
2. For index in the set $\{0, 1, \ldots, t-1\}$:
    1. If $(i < c_{\text{index}})$ : Break
    2. If $(i \& c_{\text{index}}) \neq 0$: prod $=$ prod $\circ\, \alpha_{\text{index}}$
3. Return prod

- Note that the test "the $(1 + \text{index})$-th bit in the binary representation of $i$ is 1" is identical to the test $(i \& c_{\text{index}}) \neq 0$
- If this test passes, then prod is multiplied by $\alpha_{\text{index}} = g^{2^{\text{index}}}$
- Prove: This approach correctly calculates $g^i$
- Note that the runtime is $O(\log i)$ (that is, the algorithm is efficient)

# Example Problem

- Let us consider a problem that shall use all the facts we studied about groups and fields in the last two lectures. There are multiple solutions with varying degree of complexities
- Compute

$$17^{2020} \mod 23$$

**Solution 1.**

- We can use repeated squaring directly to compute

$$
\begin{aligned}
17^1 &\mod 23 \\
17^2 &\mod 23 \\
17^4 &\mod 23 \\
&\vdots \\
17^{1024} &\mod 23
\end{aligned}
$$

- Write 2020 in binary and compute $17^{2020} \mod 23$ using the values computed above
- Although this is a correct and a tractable way to compute this value, it is computationally intensive and prone to errors (without a calculator)

**Solution 2.**

- In homework you will prove that $x^p = x \mod p$, where $p$ is a prime and $x$ is any integer
- You can use this fact to simplify the computation of $17^{2020} \mod 23$ as follows

$$
\begin{aligned}
17^{2020} \mod 23 &= 17^{23} \cdot 17^{1997} \mod 23 \\
&= \left(17^{23}\right)^{2} \cdot 17^{1974} \mod 23 \\
&\vdots \\
&= \left(17^{23}\right)^{87} \cdot 17^{19} \mod 23 \\
&= (17)^{87} \cdot 17^{19} \mod 23, \qquad \text{using } x^{p} = x \mod p \\
&= 17^{106} \mod 23 \\
&= \left(17^{23}\right)^{4} \cdot 17^{14} \mod 23 \\
&= (17)^{4} \cdot 17^{14} \mod 23, \qquad \text{using } x^{p} = x \mod p \\
&= 17^{18} \mod 23
\end{aligned}
$$

- This final expression can be computed using the repeated squaring technique

**Solution 3.**

- In homework you will prove that $x^{p-1} = 1 \mod p$, where $p$ is a prime and $x$ is any integer NOT divisible by $p$ (there are also alternate proofs of this statement by considering the size of the subgroup of $(\mathbb{Z}_p^*, \times)$ that is generated by $x$)
- So, we can compute the expression as follows

$$
\begin{aligned}
17^{2020} \mod 23 &= 17^{22} \cdot 17^{1998} \mod 23 \\
&= \left(17^{22}\right)^2 \cdot 17^{1976} \mod 23 \\
&\vdots \\
&= \left(17^{22}\right)^{91} \cdot 17^{18} \mod 23 \\
&= (1)^{91} \cdot 17^{18} \mod 23, \qquad \text{using } x^{p-1} = 1 \mod p \\
&= 17^{18} \mod 23
\end{aligned}
$$

- BTW, in general you can conclude that

$$x^n = x^{n \mod p-1} \mod p,$$

for any integer $n$ and any integer $x$ that is not divisible by $p$

- Now you can compute $17^{18} \mod 23$ result using repeated squaring technique

$$17^1 = 17 \mod 23$$
$$17^2 = 13 \mod 23$$
$$17^4 = 8 \mod 23$$
$$17^8 = 18 \mod 23$$
$$17^{16} = 2 \mod 23$$

- Now, we have

$$\begin{aligned} 17^{18} &= 17^{16+2} \mod 23 \\ &= 17^{16} \cdot 17^{2} \mod 23 \\ &= 2 \cdot 13 \mod 23 \\ &= 3 \mod 23 \end{aligned}$$

- Therefore, we conclude that

$$17^{2020} = 17^{18} = 3 \mod 23.$$

That is our answer!